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SOME RESULTS ON p^{th} GOL'DBERG RELATIVE ORDER

BALRAM PRAJAPATI & ANUPMA RASTOGI
Department of Mathematics and Astronomy University of Lucknow, Lucknow, India

ABSTRACT

In this paper, we obtain some results on the p^{th} Gol'dberg relative order of entire functions of several complex variables which improve some earlier results.

KEYWORDS: Entire Function, Relative Order, Gol'dberg Order, Gol'dberg Relative Order, Property(R)

INTRODUCTION

Let f and g be two non-constant entire functions and

$M_f(r) = \max\{|f(z)|: |z| = r\}$,$M_g(r) = \max\{|g(z)|: |z| = r\}$ be the maximum modulus functions of f and g respectively.Then $M_f(r)$ is strictly increasing function and continuous of r and its inverse $M_f^{-1}: (|f(0)|, \infty) \to (0, \infty)$ exits and$\lim_{s\to\infty} M_f^{-1}(s) = \infty$.

Definition 1: The order ρ_f and lower order λ_f of an entire function f are defined as follows:

$\rho_f=\lim_{r\to\infty}\sup\frac{\log^{[2]} M_f(r)}{\log r}$ and $\lambda_f=\lim_{r\to\infty}\inf\frac{\log^{[2]} M_f(r)}{\log r}$

The function f is said to be of regular growth if $\rho_f = \lambda_f$.

In [1] Lahiri and Banerjee considered a more general definition of order as follows:

Definition 2: [1] if p≥ 1 is a positive integer, then the p^{th} relative order of f with respect tog, denoted by $\rho_g^{[p]}(f)$ is defined as

$\rho_g^{[p]}(f)=\inf\{\mu > 0: M_f(r) < M_g(\exp^{[p-1]} r^{\mu}), \text{for all } r > r_0(\mu) > 0\}$

If p=1, g (z) =expz, then $\rho_g^{[p]}(f)=\rho_g(f)$ the classical order off.

Definition 3: [4] Let $f(z_1, z_2)$ and $g(z_1, z_2)$ are two non constant entire functions of two complex variables z_1 and z_2 holomorphic in the closed polydisc

$\{(z_1, z_2): |z_i| \le r_i; i = 1,2\}$ And

Let $M_f(r_1, r_2) = \max\{|f(z)|: |z_i| = r_i; i = 1,2\}$,$M_g(r_1, r_2) = \max\{|g(z_1, z_2)|: |z_i| \le r_i; i = 1,2\}$

The relative order of f with respect to g denoted by$\rho_g(f)$, is defined as

$\rho_g(f) = \inf\{\mu > 0: M_f(r) < M_g({r_1}^{\mu}, {r_2}^{\mu}), \text{for all } r_1 \ge R(\mu), r_2 \ge R(\mu)\}$.

We recall the following notation and definition of relative order of entire functions of n complex variables. We denote the point $(z_1, z_2, \ldots\ldots, z_n) \in \mathbb{C}^n$and $(m_1, m_2, \ldots\ldots, m_n) \in I^n$ by z and m respectively I denote the set of all non negative integers.Where $\mathbb{C}^n$denote the n-dimensional complex space. $|z| = (|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2)^{1/2}$ also we can write $||m|| = m_1 + m_2 + \cdots + m_n$.Let bounded complete $n-$ circular domain $D \subseteq \mathbb{C}^n$ with centre at origin. Let

$$M_{f,D}(R) = \sup_{z \in D_R} |f(z)|$$

For $R > 0$ a point $z \in D_R$iff $\frac{z}{R} \in D$ where f is an entire function of n complex variables Let g be a non constant entire function then $M_{g,D}(R)$ is strictly increasing continuous and its inverse

$M_{g,D}^{-1}: (|g(0)|, \infty) \to (0, \infty)$ Exists such that $\lim_{R\to\infty} M_{g,D}^{-1}(R) = \infty$.The Gol'dberg order of entire function of n complex variables is defined as follows:

Definition 4: [7] The Gol'dberg order $\rho_{f,D}$ of f with respect to domain is defined as follows

$$\rho_{f,D} = \lim_{R\to\infty}\sup \frac{\log^{[2]} M_{f,D}(R)}{\log R}$$

The lower Gol'dberg order $\lambda_{f,D}$ of f with respect to domain D is defined as

$$\lambda_{f,D} = \lim_{R\to\infty}\inf \frac{\log^{[2]} M_{f,D}(R)}{\log R}$$

If f is regular growth if $\rho_{f,D} = \lambda_{f,D}$.

The$\rho_{f,D}$, order is independent of the choice of the domain D in {cf. [7]} and therefore we denote the order of f asρ_f.

Definition 5: p^{th} Gol'dberg order and lower Gol'dberg order are denoted by $\rho_{f,D}^{[p]}$ and $\lambda_{g,D}^{[p]}$ are respectively defined as follows:

$$\rho_{f,D}^{[p]} = \lim_{R\to\infty}\sup \frac{\log^{[p]} M_{f,D}(R)}{\log R}$$

And

$$\lambda_{f,D}^{[p]} = \lim_{R\to\infty}\inf \frac{\log^{[p]} M_{f,D}(R)}{\log R}$$

Where $p = 2,3,4, \ldots$

In recent paper Mondale and Roy [6] introduced the concept of relative order of entire functions of n-complex variables. They gave the following definition.

Definition 6: ([1],[4],[6]) Let f, and g be entire functions of n complex variables and D be a bounded complete n-circular domain with centre at the origin in C^n.Then the p^{th} relative order $\rho_{g,D}^{[p]}(f)$ of f with respect to g in the domain D is defined by

$$\rho_{g,D}^{[p]}(f) = \inf\{\mu > 0: M_{f,D}(r) < M_{g,D}(\exp^{[p-1]} R^{\mu}), \text{for } R \geq R_0(\mu) > 0\}$$

$$= \lim_{R\to\infty}\sup \frac{\log^{[p]} M_{g,D}^{-1}(M_{f,D}(R))}{\log R}$$

Where $p = 1,2,3,,\ldots$

If we take $g(z) = e^{z} = e^{(z_1,z_2,\ldots,z_n)}$ and $p = 1$, then the relative Gol'dberg order $\rho_{g,D}(f)$ of f with respect to g in the domain D coincides with Gol'dberg $\rho_{f,D}$ of f with respect to domain D.

We define the p^{th} Gol'dberg relative order $\lambda_{g,D}^{[p]}(f)$ of f with respect to g in the domain as

$$\lambda_{g,D}^{[p]}(f) = \lim_{R\to\infty}\inf\frac{\log^{[p]} M_{g,D}^{-1}(M_{f,D}(R))}{\log R}$$

Where $p = 1,2,3,,\ldots$

In this paper we obtain some relationship between relative order, relative lower order, Gol'dberg order and Gol'dberg lower order, p^{th} Gol'dberg relative (order and lower order) of entire functions of several complex variables which improves some earlier results.

THEOREMS

Theorem 1: Let f and g be entire functions of n complex variables such that $0 < \lambda_f^{[p]} \leq \rho_f^{[p]}$ and $0 < \lambda_g^{[p]} \leq \rho_g^{[p]}$ .Then

$$\lambda_f^{[p]}/\rho_g^{[p]} \leq \lambda_g^{[p]}(f) \leq \min\left\{\lambda_f^{[p]}/\lambda_g^{[p]}, \rho_f^{[p]}/\rho_g^{[p]}\right\} \leq \max\left\{\lambda_f^{[p]}/\lambda_g^{[p]}, \rho_f^{[p]}/\rho_g^{[p]}\right\} \leq \rho_g^{[p]}(f) \leq \rho_f^{[p]}/\lambda_g^{[p]}$$

Proof: From the definition of p^{th}-Gol'dberg order and lower order we get arbitrary $\varepsilon > 0$ and for all large values of R then

$$M_f(R) < exp\left(\exp^{[p-1]} R^{\left(\rho_f^{[p]}+\varepsilon\right)}\right) \quad (1)$$

$$M_g(R) < exp\left(\exp^{[p-1]} R^{\left(\rho_g^{[p]}+\varepsilon\right)}\right) \quad (2)$$

$$M_f(R) > exp\left(\exp^{[p-1]} R^{\left(\lambda_f^{[p]}-\varepsilon\right)}\right) \quad (3)$$

$$M_g(R) > exp\left(\exp^{[p-1]} R^{\left(\lambda_g^{[p]}-\varepsilon\right)}\right) \quad (4)$$

Also for a sequence $\{R_n\}$ tending to infinity we get

$$M_f(R_n) > exp\left(\exp^{[p-1]} R_n^{\left(\rho_f^{[p]}-\varepsilon\right)}\right) \quad (5)$$

$$M_g(R_n) > exp\left(\exp^{[p-1]} R_n^{\left(\rho_g^{[p]}-\varepsilon\right)}\right) \quad (6)$$

$$M_f(R_n) < exp\left(\exp^{[p-1]} R_n^{\left(\lambda_f^{[p]}+\varepsilon\right)}\right) \quad (7)$$

And $M_f(R_n) < exp\left(\exp^{[p-1]} R_n^{\left(\lambda_g^{[p]}+\varepsilon\right)}\right)$ (8)

Now from the definition of p^{th} relative order we get, for arbitrary $\varepsilon_1 > 0$ and for all large values of R then

$$\rho_g^{[p]}(f) + \varepsilon_1 > \frac{\log^{[p]} M_g^{-1}(M_f(R))}{\log R}$$

Now from (5) we get for a sequence $\{R_n\}$ tending to infinity that

$$\rho_g^{[p]}(f) + \varepsilon_1 > \frac{\log^{[p]} M_g^{-1}\left(\exp\left(\exp^{[p-1]} R_n^{\left(\rho_f^{[p]}-\epsilon\right)}\right)\right)}{\log R_n}$$

$$= \frac{\log^{[p]} M_g^{-1}\left(\exp\left(\exp^{[p-1]} R_n^{\left(\frac{\rho_f^{[p]}-\varepsilon}{\rho_g^{[p]}+\varepsilon}\right)^{\rho_g^{[p]}+\varepsilon}}\right)\right)}{\log R_n}$$

$$> \frac{\log^{[p]} M_g^{-1} M_g\left(\exp^{[p-1]} R_n^{\left(\frac{\rho_f^{[p]}-\varepsilon}{\rho_g^{[p]}+\varepsilon}\right)}\right)}{\log R_n}$$

$$= \frac{\rho_f^{[p]}-\varepsilon}{\rho_g^{[p]}+\varepsilon}$$

As $\varepsilon_1 > 0$ and $\varepsilon > 0$ are arbitrary we get that

$$\rho_g^{[p]}(f) \geq \frac{\rho_f^{[p]}}{\rho_g^{[p]}} \tag{9}$$

Also from (1) we get for arbitrary $\varepsilon > 0$ and for all large values of R that

$$\frac{\log^{[p]} M_g^{-1}(M_f(R))}{\log R} < \frac{\log^{[p]} M_g^{-1}\left(\exp\left(\exp^{[p-1]} R^{\left(\rho_f^{[p]}+\varepsilon\right)}\right)\right)}{\log R}$$

$$= \frac{\log^{[p]} M_g^{-1}\left(\exp\left(\exp^{[p-1]} R^{\left(\frac{\rho_f^{[p]}+\varepsilon}{\rho_g^{[p]}-\varepsilon}\right)^{\rho_g^{[p]}-\varepsilon}}\right)\right)}{\log R}$$

Now from (6) we get for a sequence $\{R_n\}$ tending to infinity that

$$\frac{\log^{[p]} M_g^{-1}(M_f(R_n))}{\log R_n} < \frac{\log^{[p]} M_g^{-1} M_g\left(\exp^{[p-1]} R_n^{\left(\frac{\rho_f^{[p]}+\varepsilon}{\rho_g^{[p]}-\varepsilon}\right)}\right)}{\log R_n}$$

$$\lim_{R_n \to \infty} \inf \frac{\log^{[p]} M_g^{-1}(M_f(R_n))}{\log R_n} \leq \left(\frac{\rho_f^{[p]}+\varepsilon}{\rho_g^{[p]}-\varepsilon}\right)$$

As $\varepsilon > 0$ is arbitrary we have

$$\lambda_g^{[p]}(f) \leq \frac{\rho_f^{[p]}}{\rho_g^{[p]}} \tag{10}$$

Now from the definition of p^{th} relative lower order we get for arbitrary $\varepsilon_2 > 0$ and for all large values of R that

$$\lambda_g^{[p]}(f) - \varepsilon_2 < \frac{\log^{[p]} M_g^{-1}(M_f(R))}{\log R}$$

Now from (7) we get for a sequence $\{R_n\}$ tending to infinity

$$\lambda_g^{[p]}(f) - \varepsilon_2 < \frac{\log^{[p]} M_g^{-1}\left(\exp\left(\exp^{[p-1]} R_n^{\left(\lambda_f^{[p]}+\varepsilon\right)}\right)\right)}{\log R_n}$$

$$= \frac{\log^{[p]} M_g^{-1}\left(\exp\left(\exp^{[p-1]} R_n^{\left(\frac{\lambda_f^{[p]}+\varepsilon}{\lambda_g^{[p]}-\varepsilon}\right)^{\lambda_g^{[p]}-\varepsilon}}\right)\right)}{\log R_n}$$

$$< \frac{\log^{[p]} M_g^{-1} M_g\left(\exp^{[p-1]} R_n^{\left(\frac{\lambda_f^{[p]}+\varepsilon}{\lambda_g^{[p]}-\varepsilon}\right)}\right)}{\log R_n}$$

$$= \frac{\lambda_f^{[p]}+\varepsilon}{\lambda_g^{[p]}-\varepsilon}$$

As $\varepsilon_2 > 0$ and $\varepsilon > 0$ are arbitrary we obtain that

$$\lambda_g^{[p]}(f) \leq \frac{\lambda_f^{[p]}}{\lambda_g^{[p]}} \tag{11}$$

Now from (3) we get arbitrary $\varepsilon > 0$ and for all large values of R that

$$\frac{\log^{[p]} M_g^{-1}(M_f(R))}{\log R} > \frac{\log^{[p]} M_g^{-1}\left(\exp\left(\exp^{[p-1]} R^{\left(\lambda_f^{[p]}-\varepsilon\right)}\right)\right)}{\log R}$$

$$= \frac{\log^{[p]} M_g^{-1}\left(\exp\left(\exp^{[p-1]} R^{\left(\frac{\lambda_f^{[p]}-\varepsilon}{\lambda_g^{[p]}+\varepsilon}\right)^{\lambda_g^{[p]}+\varepsilon}}\right)\right)}{\log R}$$

Now from (8) we obtain for a sequence $\{R_n\}$ tending to infinity that

$$\frac{\log^{[p]} M_g^{-1}(M_f(R_n))}{\log R_n} > \frac{\log^{[p]} M_g^{-1} M_g\left(\exp^{[p-1]} R_n^{\left(\frac{\lambda_f^{[p]}-\varepsilon}{\lambda_g^{[p]}+\varepsilon}\right)}\right)}{\log R_n}$$

$$\lim_{R_n\to\infty}\sup\frac{\log^{[p]} M_g^{-1}(M_f(R_n))}{\log R_n} \geq \frac{\lambda_f^{[p]}-\varepsilon}{\lambda_g^{[p]}+\varepsilon}$$

As $\varepsilon > 0$ is arbitrary $\varepsilon_3 > 0$ and for a sequence $\{R_n\}$ tending to infinity that

$$\rho_g^{[p]}(f) - \varepsilon_3 < \frac{\log^{[p]} M_g^{-1}(M_f(R_n))}{\log R_n} < \frac{\log^{[p]} M_g^{-1}\left(\exp\left(\exp^{[p-1]} R_n^{\left(\rho_g^{[p]}-\varepsilon\right)}\right)\right)}{\log R_n}$$

$$= \frac{\log^{[p]} M_g^{-1}\left(\exp\left(\exp^{[p-1]} R_n^{\left(\frac{\rho_f^{[p]}+\varepsilon}{\lambda_g^{[p]}-\varepsilon}\right)^{\lambda_g^{[p]}-\varepsilon}}\right)\right)}{\log R_n}$$

$$< \frac{\log^{[p]} M_g^{-1} M_g\left(\exp^{[p-1]} R_n^{\left(\frac{\rho_f^{[p]}+\varepsilon}{\lambda_g^{[p]}-\varepsilon}\right)}\right)}{\log R_n}$$

$$= \frac{\rho_f^{[p]}+\varepsilon}{\lambda_g^{[p]}-\varepsilon}$$

$\varepsilon_3 > 0$ And $\varepsilon > 0$ are arbitrary, we have

$$\rho_g^{[p]}(f) \leq \frac{\rho_f^{[p]}}{\lambda_g^{[p]}} \tag{13}$$

As we get for arbitrary $\varepsilon_4 > 0$ and $\varepsilon > 0$ for a $\{R_n\}$ tending to infinity that

$$\lambda_g^{[p]}(f) + \varepsilon_4 > \frac{\log^{[p]} M_g^{-1}(M_f(R_n))}{\log R_n} > \frac{\log^{[p]} M_g^{-1}\left(\exp\left(\exp^{[p-1]} R_n^{\left(\lambda_g^{[p]}-\varepsilon\right)}\right)\right)}{\log R_n}$$

$$= \frac{\log^{[p]} M_g^{-1}\left(\exp\left(\exp^{[p-1]} R_n^{\left(\frac{\lambda_f^{[p]}-\varepsilon}{\rho_g^{[p]}+\varepsilon}\right)^{\rho_g^{[p]}+\varepsilon}}\right)\right)}{\log R_n}$$

$$< \frac{\log^{[p]} M_g^{-1} M_g\left(\exp^{[p-1]} R_n^{\left(\frac{\lambda_f^{[p]}-\varepsilon}{\rho_g^{[p]}+\varepsilon}\right)}\right)}{\log R_n}$$

$$= \frac{\lambda_f^{[p]}-\varepsilon}{\rho_g^{[p]}+\varepsilon}$$

As $\varepsilon_4 > 0$ and $\varepsilon > 0$ are arbitrary, we obtain that

$$\lambda_g^{[p]}(f) \geq \frac{\lambda_f^{[p]}}{\rho_g^{[p]}} \tag{14}$$

The theorem follows from (9), (10), (11), (12), (13) and (14).

Corollary 1: If both f and g are regular growth with non zero order then

$\rho_g^{[p]}(f) = \rho_f^{[p]}(g)$ iff $\rho_g^{[p]} = \rho_f^{[p]}$.

Corollary 2: If both f and g are regular growth with non zero order then

$\lambda_g^{[p]}(f) = \rho_g^{[p]}(f)$

Corollary 3: If g is regular growth with non zero order then

$$\rho_g^{[p]}(f) \geq \frac{\rho_f^{[p]}}{\rho_g^{[p]}}$$

Theorem 2: Let f and g be entire functions of n complex variables such that $\rho_f^{[p]} = 0$ and $0 < \rho_g^{[p]} < \infty$.Then $\lambda_g^{[p]}(f) = 0$

Proof: From the definition of Gol'dberg order we have for arbitrary $\varepsilon > o$ and for all large values of r that

$$M_f(R) < exp\left(\exp^{[p-1]} R^{(\varepsilon)}\right)$$

$$\frac{\log^{[p]} M_g^{-1}(M_f(R))}{\log R} < \frac{\log^{[p]} M_g^{-1}\left(\exp\left(\exp^{[p-1]} R^{(\varepsilon)}\right)\right)}{\log R}$$

$$= \frac{\log^{[p]} M_g^{-1}\left(\exp\left(\exp^{[p-1]} R^{\left(\frac{\varepsilon}{\rho_g^{[p]}-\varepsilon}\right)^{\rho_g^{[p]}-\varepsilon}}\right)\right)}{\log R}$$

Now from (6) we get for a sequence $\{R_n\}$ tending to infinity that

$$\frac{\log^{[p]} M_g^{-1}(M_f(R_n))}{\log R_n} < \frac{\log^{[p]} M_g^{-1} M_g\left(\exp^{[p-1]} R_n^{\left(\frac{\varepsilon}{\rho_g^{[p]}-\varepsilon}\right)}\right)}{\log R_n}$$

$$\lim_{R_n \to \infty} \inf \frac{\log^{[p]} M_g^{-1}(M_f(R_n))}{\log R_n} \leq \frac{\varepsilon}{\rho_g^{[p]}-\varepsilon}$$

As $\varepsilon > 0$ is arbitrary it follows that $\lambda_g^{[p]}(f) = 0$.

Theorem 3: Let f and g be entire functions of n complex variables such that $0 < \rho_f^{[p]} < \infty$ and $\rho_g^{[p]} = 0$. Then $\rho_g^{[p]}(f) = \infty$

Proof: From the definition of relative order we get for arbitrary $\varepsilon_1 > 0$ and for all large values of R that

$$\rho_g^{[p]}(f) + \varepsilon_1 > \frac{\log^{[p]} M_g^{-1}(M_f(R))}{\log R}$$

Now from (5) we get for a sequence $\{R_n\}$ tending to infinity

$$\rho_g^{[p]}(f) + \varepsilon_1 > \frac{\log^{[p]} M_g^{-1}\left(\exp\left(\exp^{[p-1]} R_n^{\left(\rho_f^{[p]}-\varepsilon\right)}\right)\right)}{\log R_n}$$

$$= \frac{\log^{[p]} M_g^{-1}\left(\exp\left(\exp^{[p-1]} R_n^{\left(\frac{\rho_f^{[p]}-\varepsilon}{\varepsilon}\right)^{\varepsilon}}\right)\right)}{\log R_n}$$

$$< \frac{\log^{[p]} M_g^{-1} M_g\left(\exp^{[p-1]} R_n^{\left(\frac{\rho_f^{[p]}-\varepsilon}{\varepsilon}\right)}\right)}{\log R_n}$$

$$= \frac{\rho_f^{[p]}-\varepsilon}{\varepsilon}$$

As $\varepsilon_1 > 0$ and $\varepsilon > 0$ are arbitrary it follows that $\rho_g^{[p]}(f) = \infty$.

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